Last Time

- What is viscosity?

$$\eta \sim \frac{T}{\sigma_0} \sim \frac{T^3}{\alpha_s^2} \qquad\qquad \frac{\eta}{e+p} \sim \left\langle v_{th}^2 \right\rangle \tau_R \sim \left\langle v_{th} \right\rangle \ell_{\mathrm{m.f.p.}}$$

- Estimate of viscosity at $\tau_0 \approx 1$ fm

$$\Gamma_s \equiv \frac{\eta}{e+p} \sim \text{A few} \times \frac{1}{2\pi T}$$

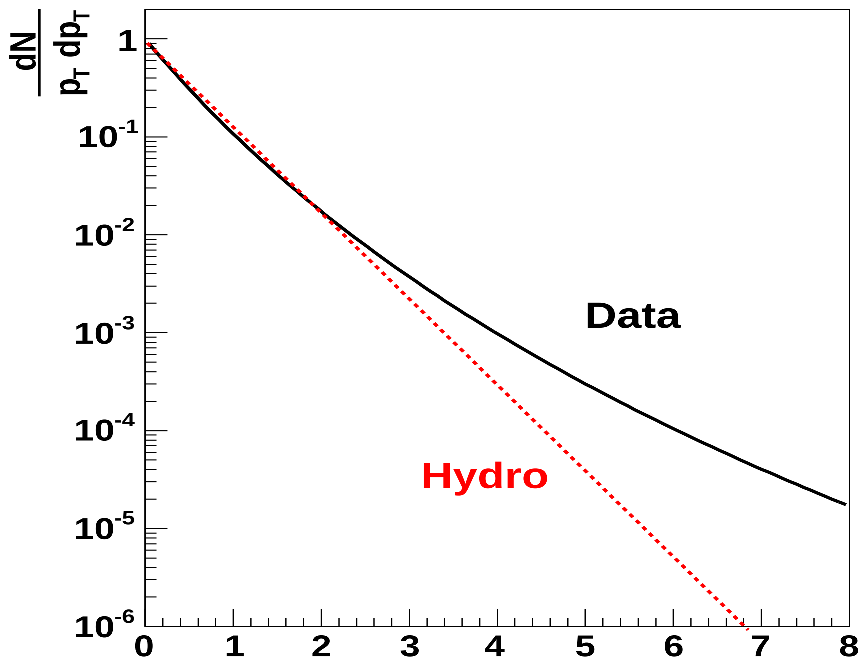
- The relevant quantity is mean free path by expansion rate:

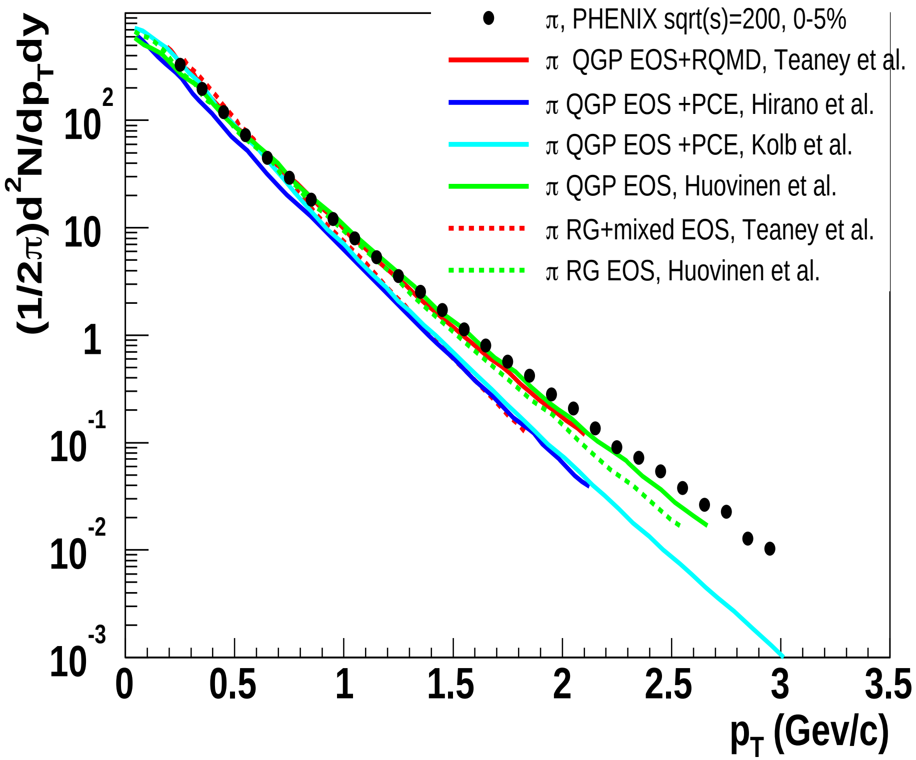
$$\frac{\Gamma_s}{\tau} \sim 1 \div \frac{1}{10}$$

- The pressure is reduced in the longitudinal direction:

$$T^{zz} = p - \frac{4}{3}\frac{\eta}{\tau}$$

Spectra



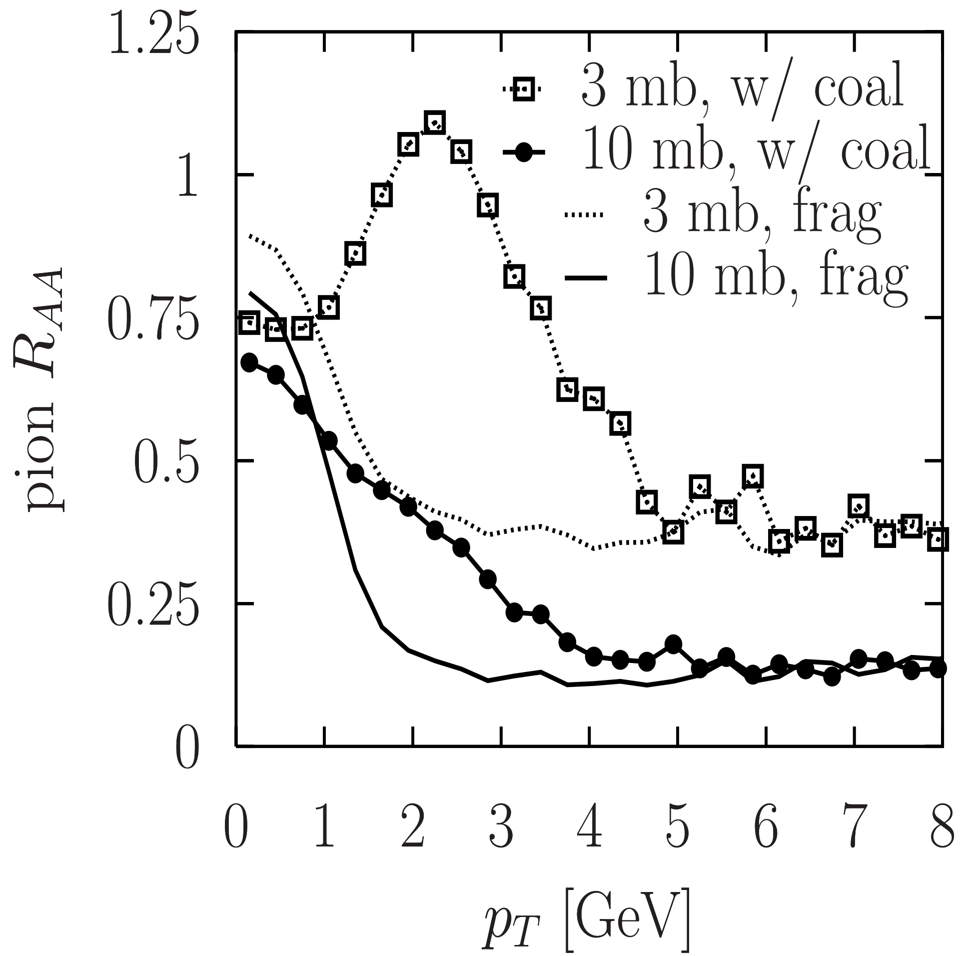


Where does hydro stop?

- Viscosity: start from below and work up
- Energy Loss: start from above and work down

Constraints on η from Energy Loss: working down

- Classical Boltzman Simulations by Molnar $N = 1000$ and $\sigma_0 = 10$ mb



This puts a bound on the viscosity:

- $\sigma_0 \lesssim 10\,\text{mb}$
- Compare

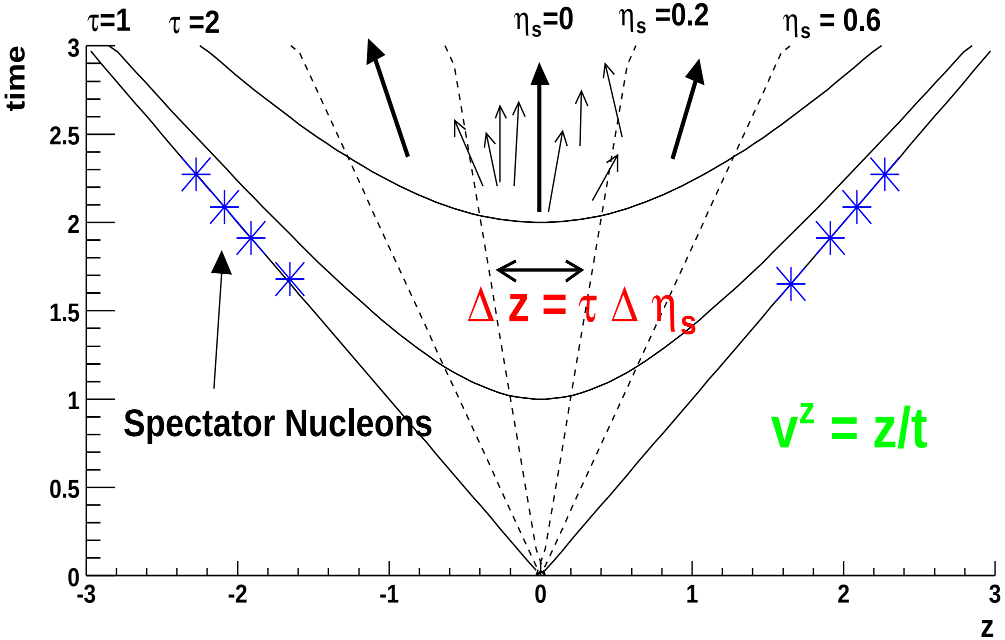
$$\frac{1}{n\sigma_0} = 0.1\,\text{fm}\left(\frac{10\,\text{mb}}{\sigma_0}\right)\left(\frac{1000}{N}\right)\left(\frac{A}{100\,\text{fm}^2}\right)\left(\frac{\tau}{1\,\text{fm}}\right)$$

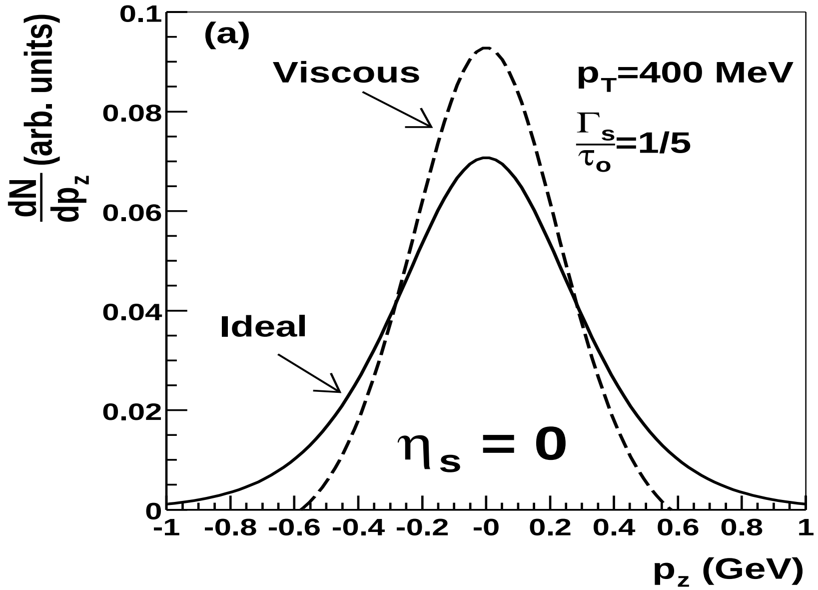
$$\frac{\eta}{e+p} = 0.1\,\text{fm}\left(\frac{\eta/s}{0.1}\right)\left(\frac{200\,\text{MeV}}{T}\right)$$

- Modelling to get from high p_T to low p_T

$$\eta/s \gtrsim 0.1$$

Working up: Thermal Spectra





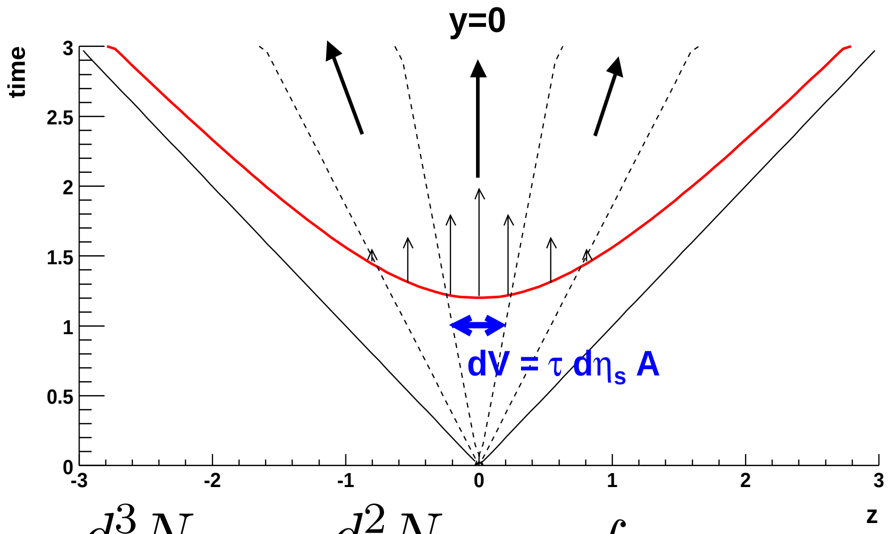
In equilibrium the thermal distribution is

$$f_0 = \frac{1}{e^{p_\alpha U^\alpha / T} - 1} = \frac{1}{e^{m_T \cosh(y - \eta_s)} - 1} \to \frac{1}{e^{E/T} - 1}$$

The effect of the viscosity is to reduce the longitudinal pressure.

$$T^{zz} = p - \frac{4}{3}\frac{\eta}{\tau} = \int d^3p \frac{p^z\, p^z}{E}(f_0 + \delta f)$$

Thermal Transverse Momentum Spectra at Mid Rapidity:



$$E\frac{d^3N}{d^3p} = \frac{d^2N}{d^2p_T\,dy} = \int \underbrace{p^\mu d\Sigma_\mu}_{E\,dV} f$$

Lets compute this integral:

$$\begin{aligned}
\frac{dN}{d^2p_T\,dy} &= \int dV\; m_T \cosh(\eta_s)\, e^{-pu/T} \\
&= \int A\,\tau d\eta_s\;\; m_T \cosh(\eta_s)\; e^{-\frac{m_T}{T}\cosh(\eta_s)} \\
&= (A\,\tau)\; m_T\;\; 2\,K_1\left(\frac{m_T}{T}\right)
\end{aligned}$$

Want to calculate δf: Use the linearized Boltzmann equation

$$\frac{p^\mu}{E}\partial_\mu f_p = \int_{1,2,3} d\Gamma_{12\to 3p}\left(f_1 f_2 - f_3 f_p\right)$$

Linearize the Boltzmann equation:

- Substitute $f \to f^e + \delta f$ with $f_p^e = e^{-pu/T}$
- Keep first order in gradients.
- Use equilibrium: $f_1^e f_2^e = f_3^e f_4^e$

$$\frac{p^\mu}{E}\partial_\mu f_p^e = \int_{1,2,3} d\Gamma_{12\to 3p}\, f_1^e f_2^e \left[\frac{\delta f_1}{f_1^e} + \frac{\delta f_2}{f_2^e} - \frac{\delta f_3}{f_3^e} - \frac{\delta f_4}{f_4^e}\right]$$

This is an integral equation for δf.

Guess the solution to the integral equation

- δf is proportional to the strains:
  $\langle \nabla_\mu u_\nu \rangle$, $\nabla_\mu u^\mu$, $\nabla_\mu T$.
- δf is a scalar $\quad \delta f \propto \chi(p) p^\mu p^\nu \langle \partial_\mu u_\nu \rangle$.
- If I restrict $f(p) = f_o(1 + \text{g(p)})$ where g(p) is a polynomial of degree less than three, the form is completely determined:

$$f = f_o(\frac{p \cdot u}{T}) \left(1 + \frac{C}{T^3} p_\mu p_\nu \frac{\langle \partial^\mu u^\nu \rangle}{2} \right)$$

- This is sometimes called the first approximation
- It is equivalent to a p_T dependent relaxation time approximation.

Full analysis

$$\frac{p^\mu}{E}\partial_\mu f_p^e = \int_{1,2,3} d\Gamma_{12\to 3p}\, f_1^e f_2^e \left[\frac{\delta f_1}{f_1^e} + \frac{\delta f_2}{f_2^e} - \frac{\delta f_3}{f_3^e} - \frac{\delta f_4}{f_4^e}\right]$$

Which gradients actually appear? $\partial_\mu = -u_\mu D + \nabla_\mu$

$$p^\mu \partial_\mu \left(e^{-pu/T}\right) \quad = \quad -f^e \left[\underbrace{-(p\cdot u)\,(\frac{p}{T}\cdot Du)}_{1} - (p\cdot u)^2 D\left(\frac{1}{T}\right) + \frac{p^\mu p^\alpha}{T}\nabla_\mu u_\nu + \underbrace{(p\cdot u)\,(p\cdot\nabla\left(\frac{1}{T}\right))}_{1}\right]$$

- Use ideal EOM $Du = -\frac{\nabla p}{e+p}$ then find

$$\underbrace{\cdots}_{1} \propto \frac{1}{T}\frac{\nabla p}{e+p} + \nabla\left(\frac{1}{T}\right) = \frac{n}{e+p}\nabla(\mu/T) = 0$$

- $D(1/T) \propto De$. Then use ideal EOM $De = -(e+p)\nabla_\mu u^\mu$

$$-(p\cdot u)^2 D\left(\frac{1}{T}\right) = \frac{(p\cdot u)^2}{T}\,\frac{e+p}{Tc_v}\,\nabla_\mu u^\mu$$

‘

Put it all together:

$$p^\mu \partial_\mu f^e \quad = \quad -f^e \left[\underbrace{\left(-\frac{(p\cdot u)^2}{T}\,\frac{e+p}{Tc_v} + \frac{1}{3}\,\frac{p\cdot\Delta\cdot p}{T} \right)} \nabla_\mu u^\mu + \frac{p^\mu p^\alpha}{T}\,\langle \nabla_\mu u_\alpha \rangle \right] = C[\delta f]$$

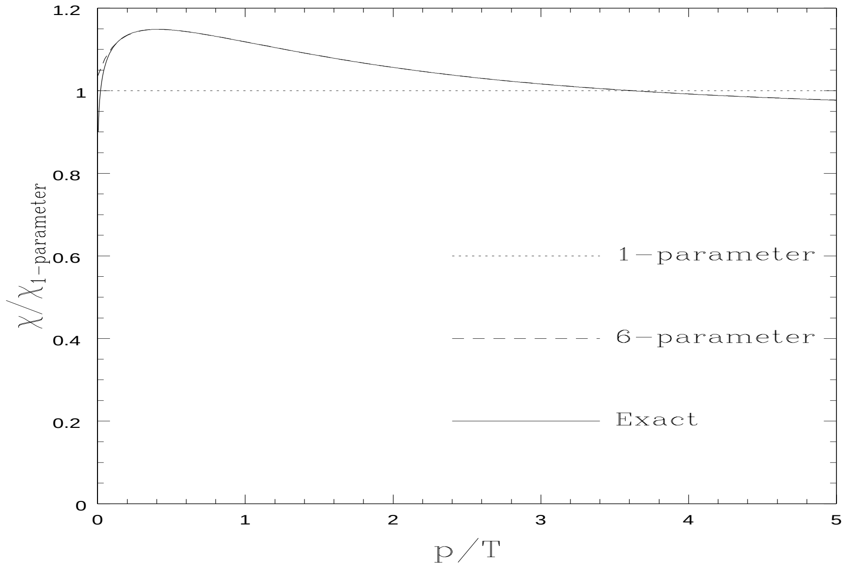
- Look at the bulk viscosity. For a massless ideal gas we have:

$$\epsilon \propto T^4 \text{ and } Tc_v = 4\,e \text{ and } e+p = \frac{4}{3}e \qquad \Longrightarrow \qquad \underbrace{\cdots} = 0$$

  - The bulk viscosity vanishes for a scale invariant ultra-relativistic gas.
  - It also vanishes for a non-relativistic Boltzmann gas

- The form of the shear correction motivates the polynomial ansatz taken before.

$$f = f_o(\frac{p\cdot u}{T}) \left(1 + \frac{C}{T^3} p_\mu p_\nu \frac{\langle \partial^\mu u^\nu \rangle}{2} \right)$$



$$f = f_o\left(\frac{p \cdot u}{T}\right)\left(1 + \frac{C}{T^3} p_\mu p_\nu \frac{\langle \partial^\mu u^\nu \rangle}{2}\right)$$

The constant $\frac{C}{T}$ is basically η/s:

$$T^{\mu\nu} = \int d^3p \, \frac{p^\mu p^\nu}{E} \, f$$

$$T_o^{\mu\nu} + T_{vis}^{\mu\nu} = \int d^3p \, \frac{p^\mu p^\nu}{E} \, (f_o + \delta f)$$

Then looking only at the viscous piece:

$$T_{vis}^{\mu\nu} = \eta \, \langle \partial^\mu u^\nu \rangle = \underbrace{\int d^3p \, \frac{p^\mu p^\nu}{E} \, f_o \, \frac{C}{T^3} p_\alpha p_\beta}_{C = \frac{\eta}{s} \text{ for a classical gas}} \frac{\langle \nabla_\alpha u_\beta \rangle}{2}$$

Viscous corrections to p_T spectrum

$$dN_o + \delta\, dN \quad = \quad \int p^\mu d\Sigma_\mu\, f_o + \delta f$$

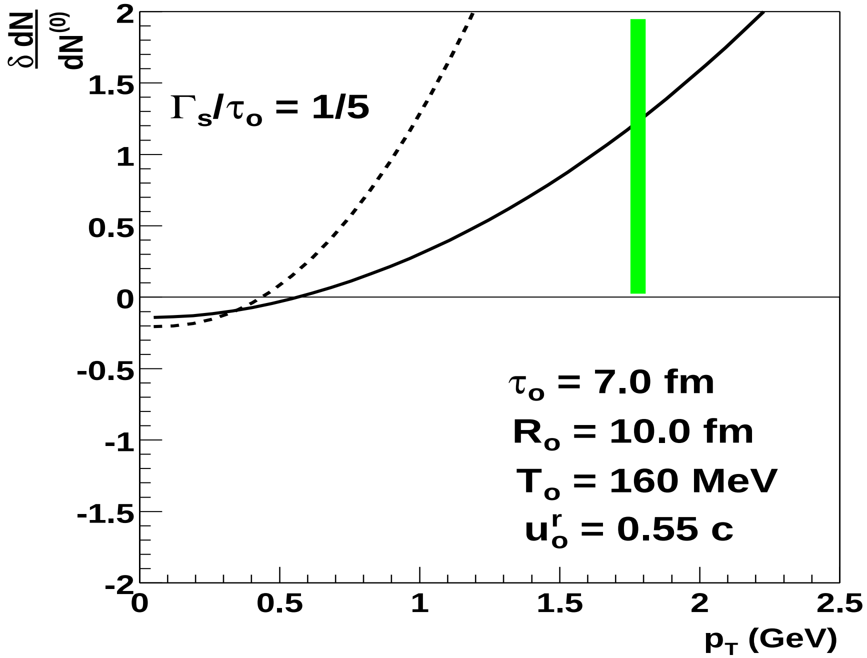
Want to compute $\frac{\delta\, dN}{dN_o}$:

$$\delta f = f_0\, \Gamma_s \frac{p_\alpha}{T} \frac{p_\beta}{T} \left\langle \nabla^\alpha u^\beta \right\rangle \sim f_0 \left(\frac{p_T}{T}\right)^2 \frac{2}{3} \frac{\Gamma_s}{\tau}$$

Now you can do these integrals:

$$\begin{aligned} \frac{\delta\, dN}{dN_o} \quad &= \quad \frac{\Gamma_s}{4\tau} \left\{ \left(\frac{p_T}{T}\right)^2 - \left(\frac{m_T}{T}\right)^2 \frac{1}{2} \left(\frac{K_3(\frac{m_T}{T})}{K_1(\frac{m_T}{T})} - 1 \right) \right\} \\ &\rightarrow \quad \frac{\Gamma_s}{4\tau} \left(\frac{p_T}{T}\right)^2 \end{aligned}$$

Viscous corrections grow with p_T

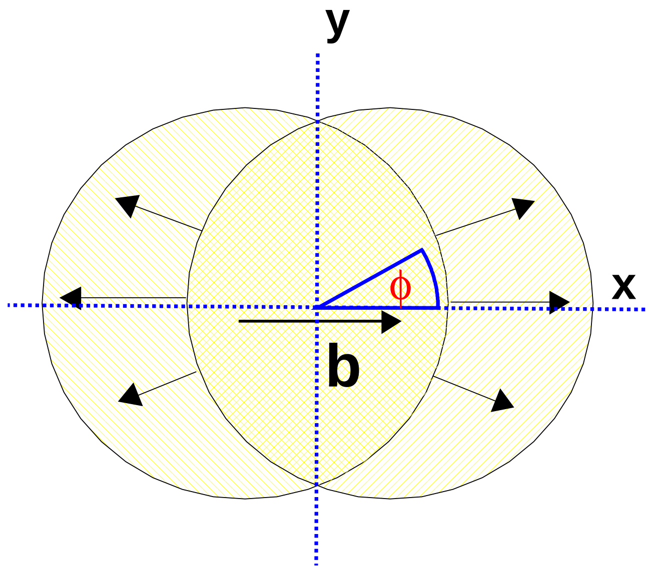


- When viscous corrections become of order one we must stop hydrodynamics.
- Viscosity puts a bound on how high in p_T the hydrodynamics may be applied
- For this room: $\frac{\Gamma_s}{\tau} \approx 10^4$ and $\frac{p_T^{\max}}{T} \approx 10^2$. Now $n \sim e^{-p/T}$

You can't see the end!

- For heavy ion collisions: $T \approx 200$ MeV find $p_T^{\max} \approx 1$ GeV.

Elliptic Flow in Heavy Ion Collisions: Qualitative



Measure the Anisotropy:

$$\frac{dN}{d\phi} = N(1 + 2\,v_2\cos(2\phi) + \cdots)$$

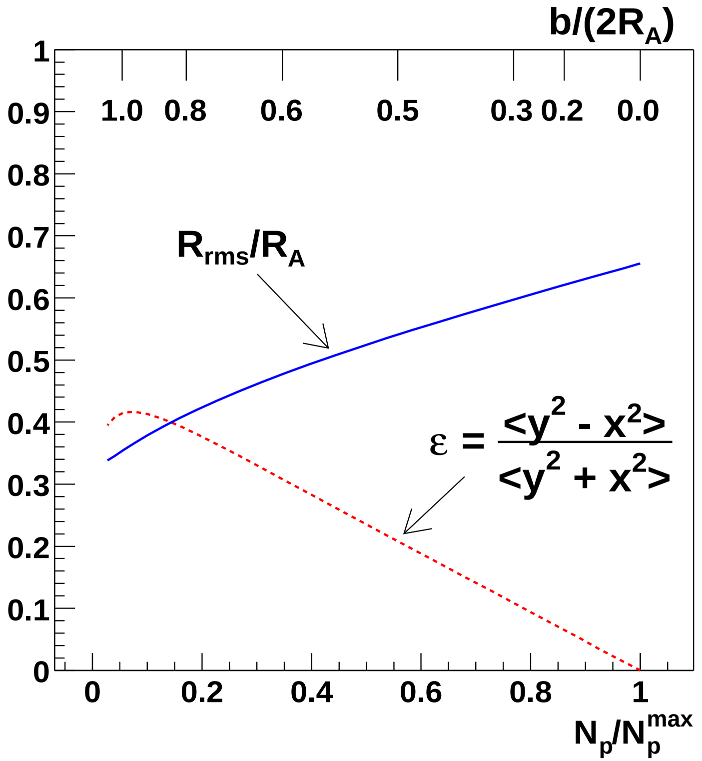
$$v_2 = \langle\cos(2\phi)\rangle$$

Can also bin in p_T:

$$\frac{dN}{p_T dp_T d\phi} = N(1 + 2\,v_2(p_T)\cos(2\phi) + \cdots)$$

$$v_2 = \langle\cos(2\phi)\rangle_{p_T}$$

Categorize the collision geometry:



1. $N_p \equiv$ The number of $\overbrace{\text{participating nucleons}}^{\text{participants}}$.

2. $R\text{rms} \equiv \sqrt{\langle x^2 + y^2 \rangle}$. The size of the collision zone.

3. $\epsilon \equiv$ The anisotropy of the initial geometry

Facts:

1. $\frac{dN}{dy} \propto N_p =$ the number of participants

2. $\epsilon \propto N_p =$ the number of participants nucleons.

3. Centrality $\approx \left(\frac{b}{2\,R_A}\right)^2$. Example $16 - 24\%$ central is $b \approx 7\,fm$

Basic Analysis of Elliptic Flow:

- Since ϵ is small we expect:

$$v_2 \propto \epsilon \propto 1 - N_p/N_p^{max}$$

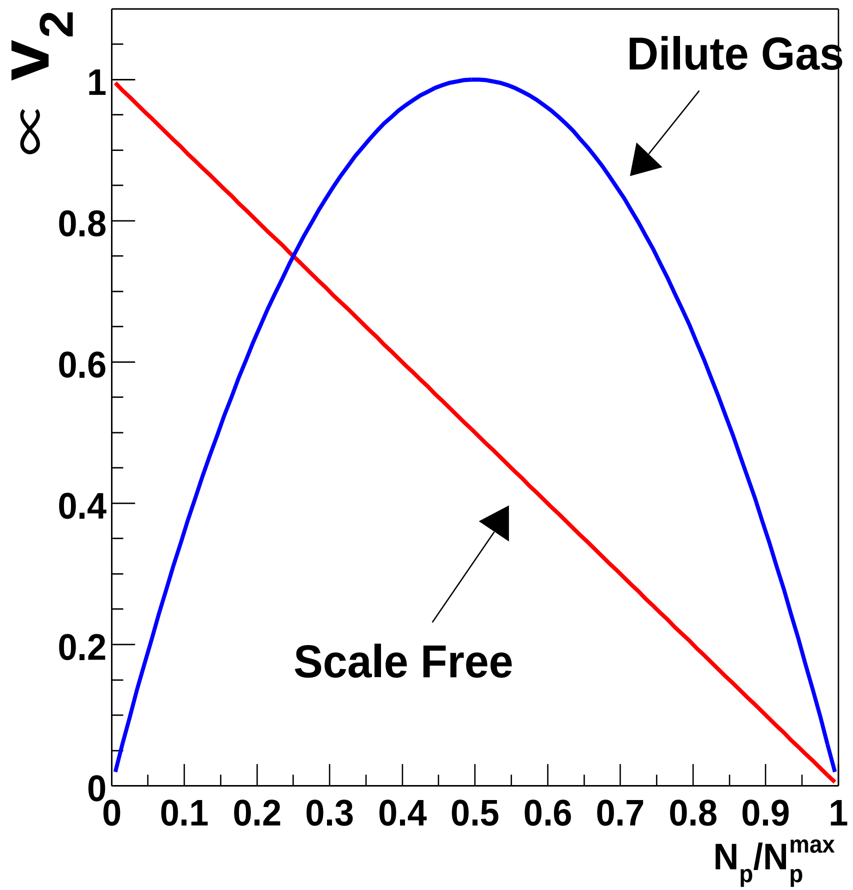
- For a system with no other scales in the problem, the physics is independent of centrality

$$v_2 = \text{Const} \times (1 - N_p/N_p^{max})$$

  Ideal hydrodynamics has no scales and the response is essentially trivially related to geometry.

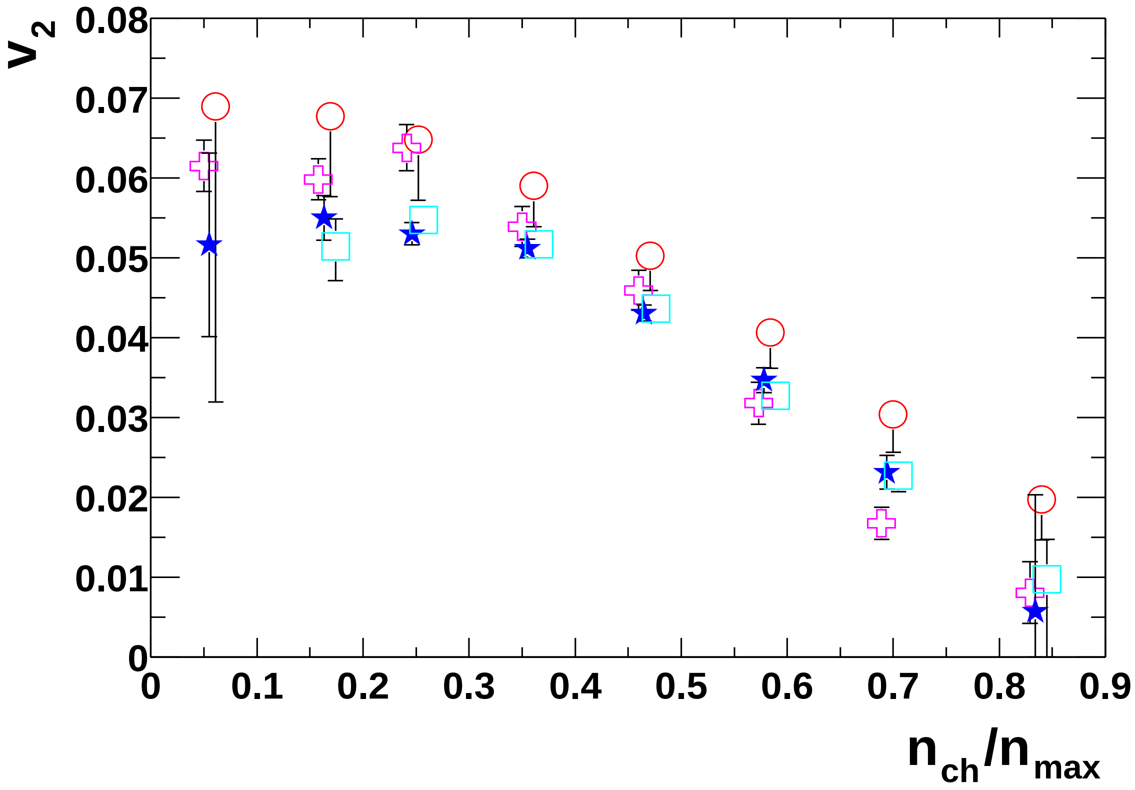
- For a dilute system (with constant cross sections) we expect collective response to be proportional to multiplicity $v_2 \propto \frac{dN}{dy} \propto N_p$.

$$v_2 \propto N_p(1 - N_p/N_p^{max})$$



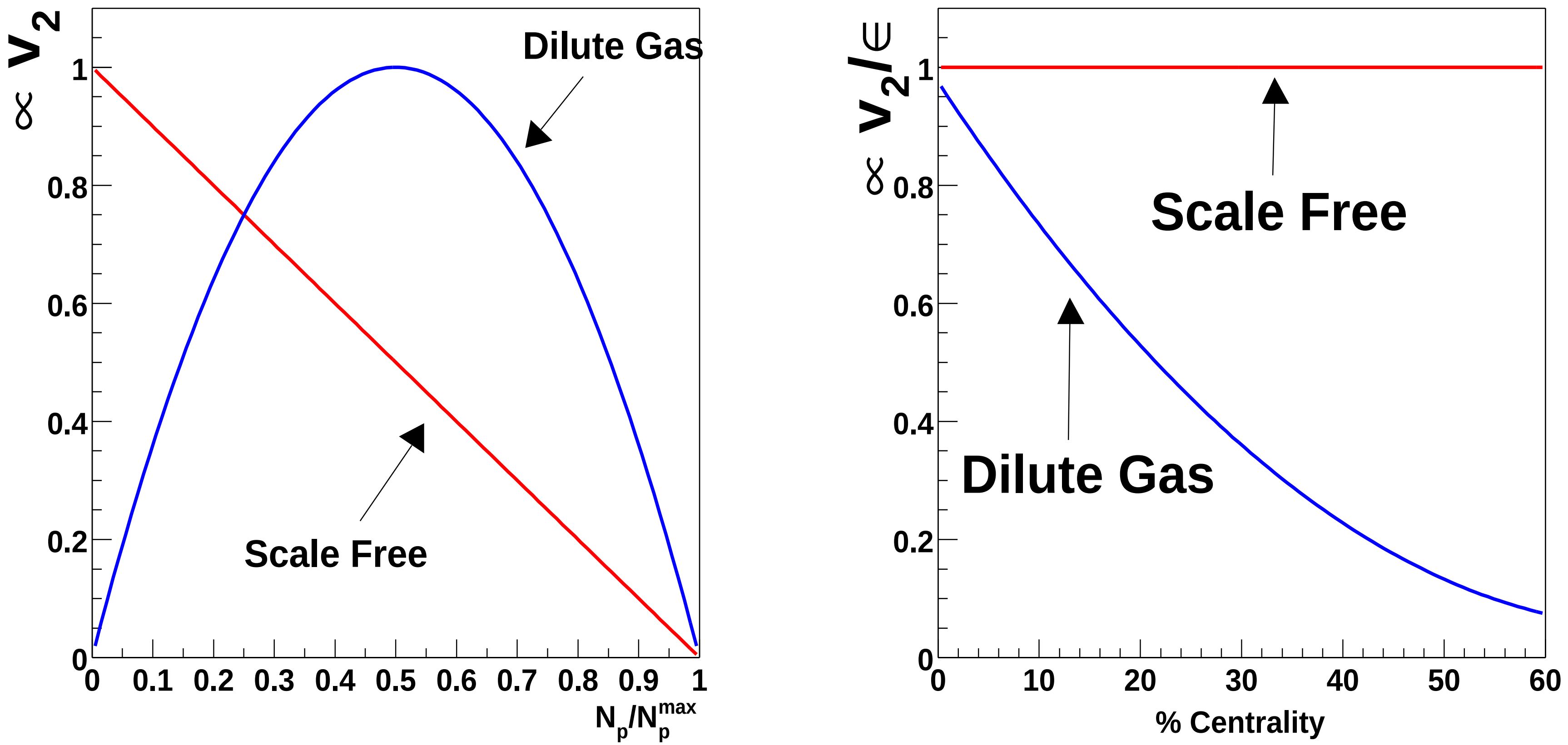
- Viscous hydrodynamics is in-between these two cases.

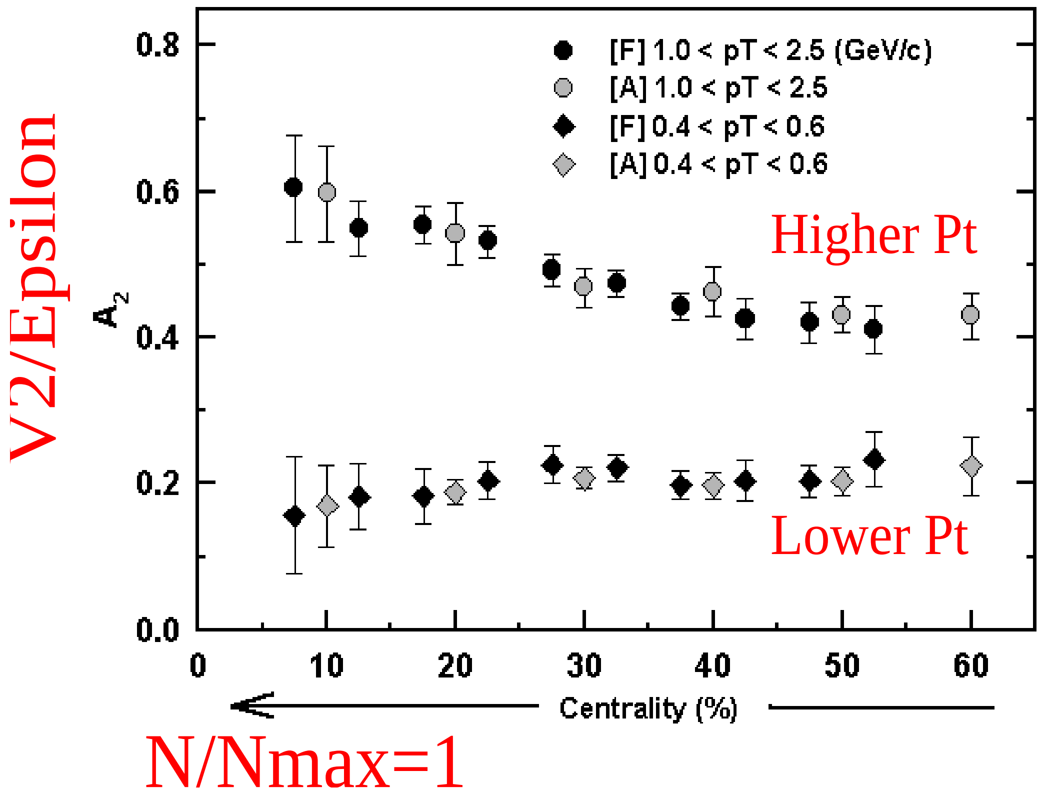
Observation of v_2 at RHIC



- If nothing changes as a function of centrality then expect: $v_2 \propto \epsilon$
- Up to corrections: $v_2 \propto \epsilon$ in data

Translation

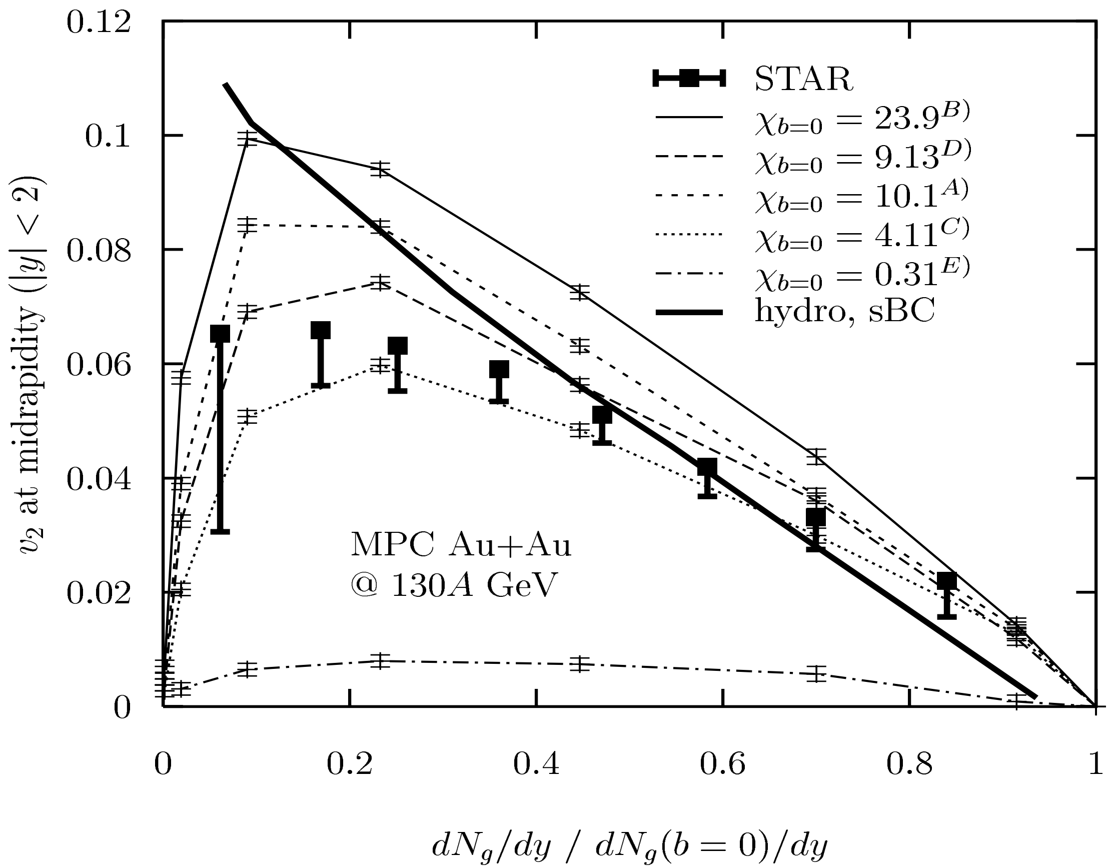




- At lower $p_T \approx 0.6$ GeV the response is directly proportional to ϵ
- At higher $p_T \approx 1.4$ GeV the effects of other scales come in.

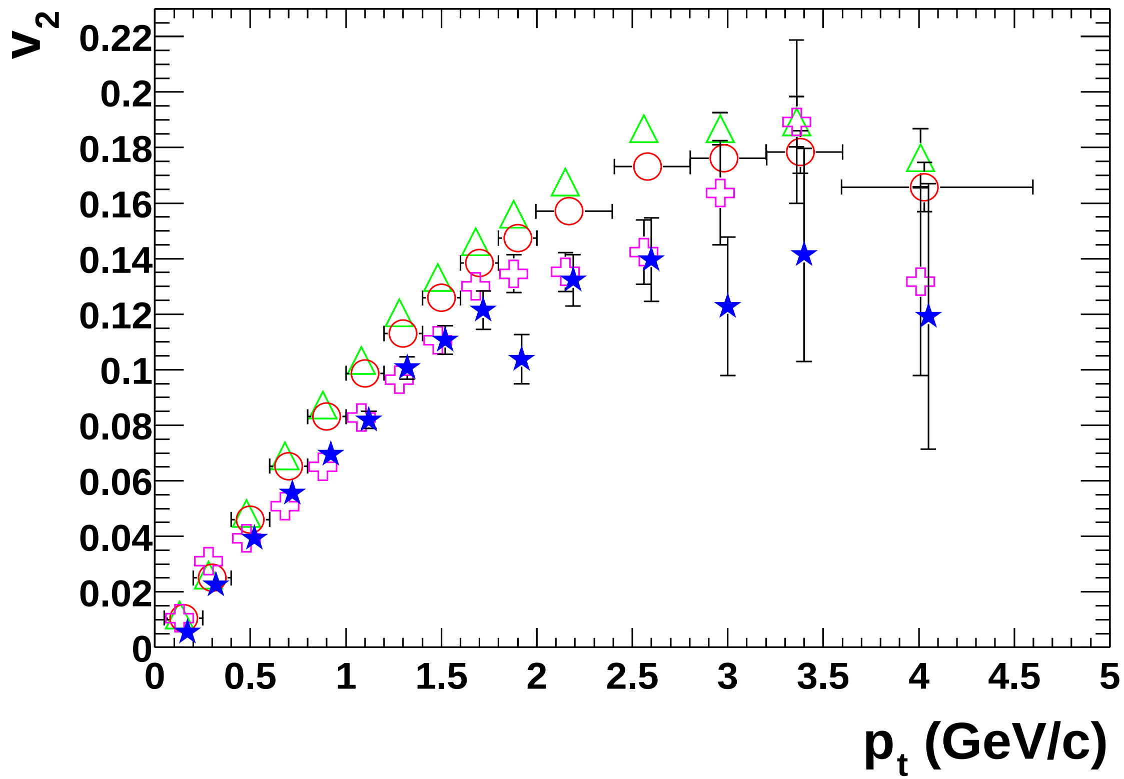
Beware non-flow! This is improtant to settle

Solution to Boltzmann Equation: (Molnar & Kolb)



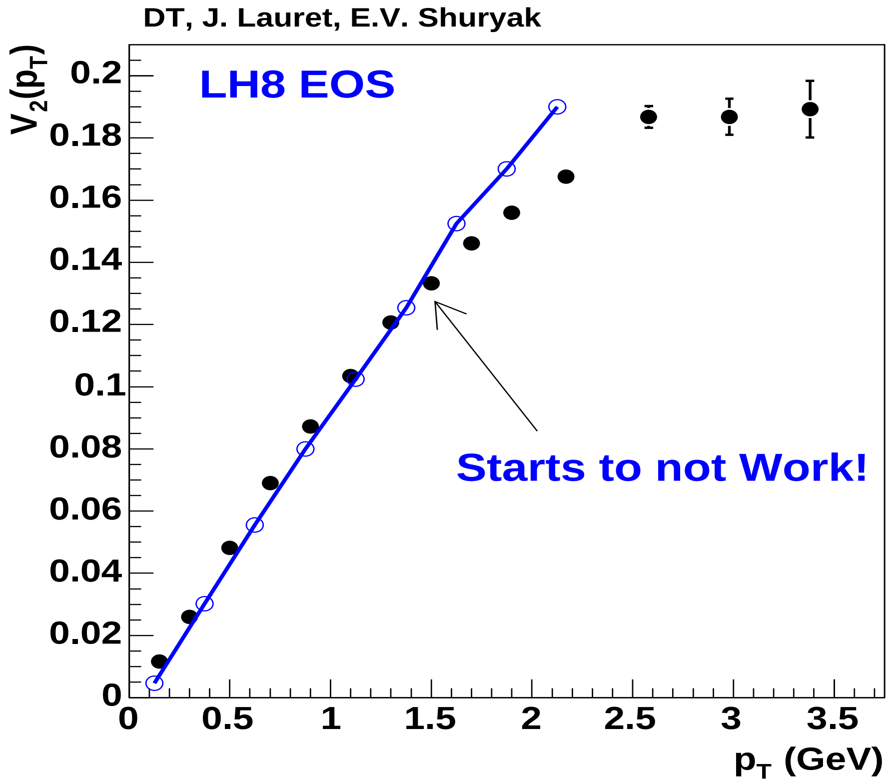
- $\chi_{b=0} = 10$ corresponds to $(\Gamma_s/\tau)_0 \approx 0.04$
- For the Boltzmann equation, v_2 curves over in peripheral collisions.

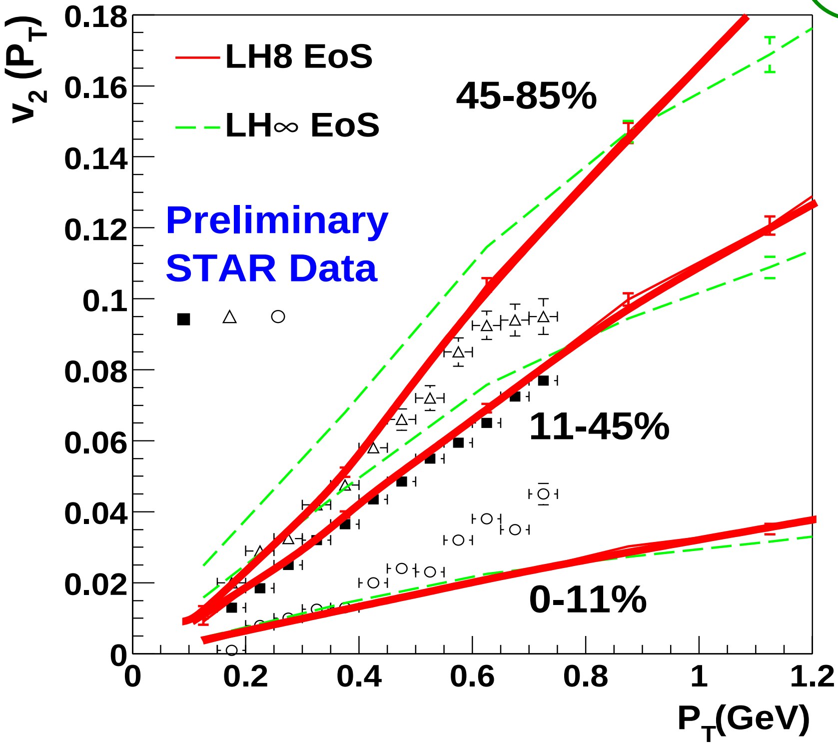
v_2 as a Function of Transverse Momentum:

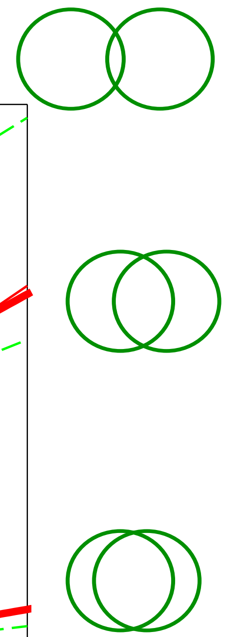


- $v_2(p_T)$ increases until $p_T \approx 2.0$ GeV and then flattens.
- v_2 is large even at $p_T \approx 3.0$ GeV.
- There is a 1.7 to 1 asymmetry between x and y at $p_T = 3.0$ GeV.

Comparison with Hydrodynamic Models



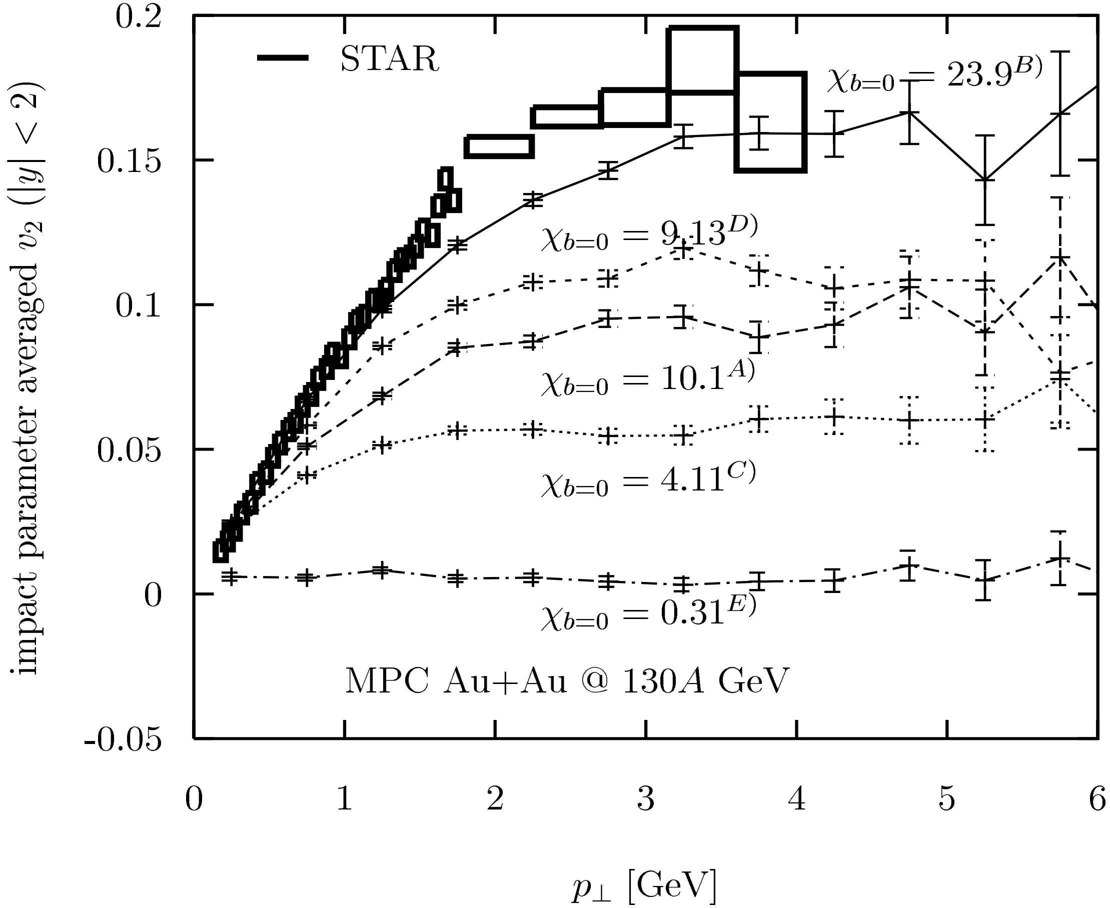




- Can account for the magnitude of v_2 and dependence on centrality – roughly
- Can account for the linear rise but not for the saturation of v_2 at moderate momenta

Comparison with the Boltzmann Equation: Denes Molnar + M. Gyulassy

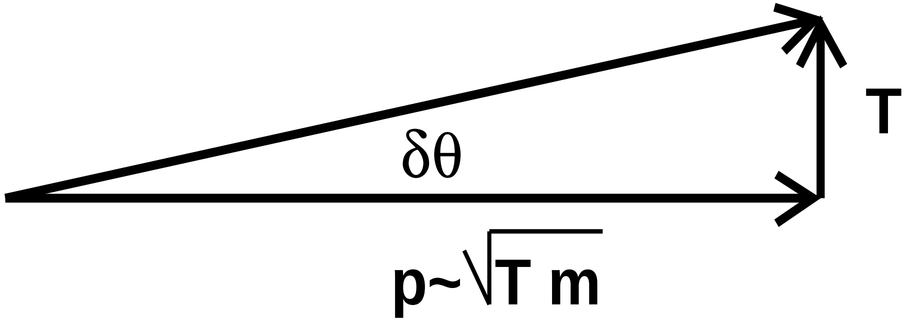
- Classical Massless Particles with Constant Cross Sections



- The Boltzmann equation predicted a flattening of v_2 at high p_T
- The observed $v_2(p_T)$ is consistent with viscous/Boltzmann effects.

Langevin and Heavy Quarks

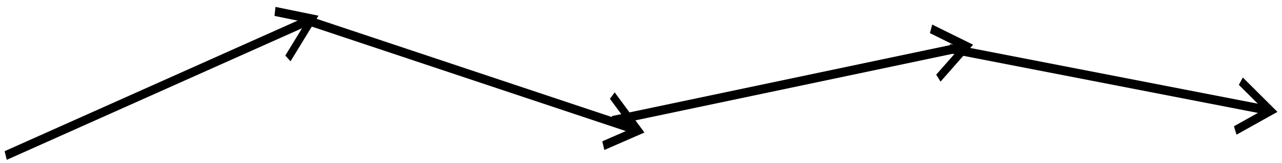
- A tool to study elliptic flow



$$\delta\theta \sim \sqrt{\frac{T}{m}}$$

- The collision only scarcely changes the direction of the charm quark

- The charm quark undergoes a random walk suffering many collisions provided $\ell_{m.f.p} \ll L$



$$(\Delta\theta)^2 \sim N_{kick}\,(\delta\theta)^2 \sim N_{kick}\,\frac{T}{m}$$

Langevin description of heavy quark thermalization:

- Write down an equation of motion for the heavy quarks.

$$\frac{dp}{dt} = -\eta_D p + \xi(t)$$

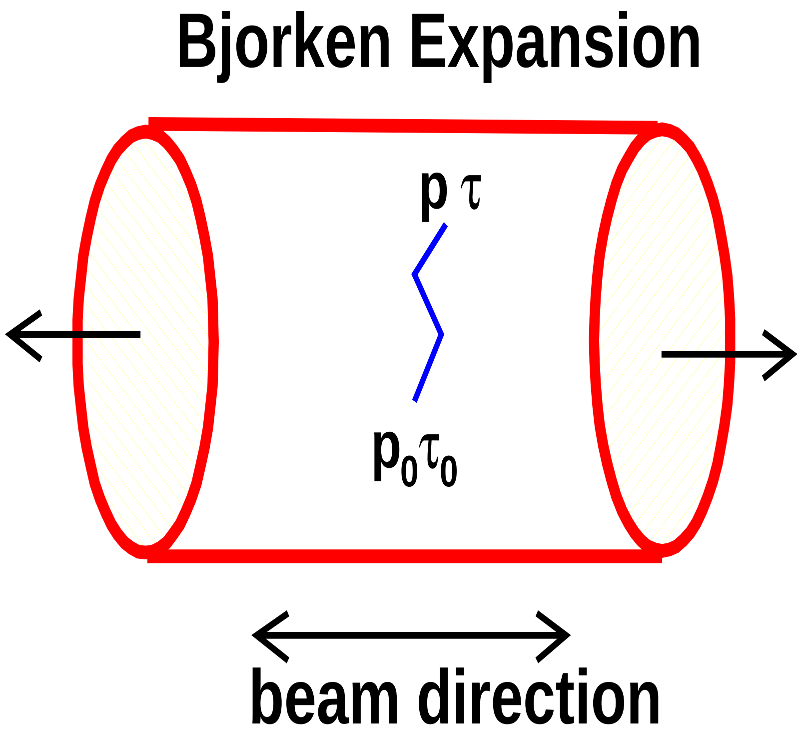
- When the number of kicks is large we replace the kicks by random kicks: $\xi(t)$.

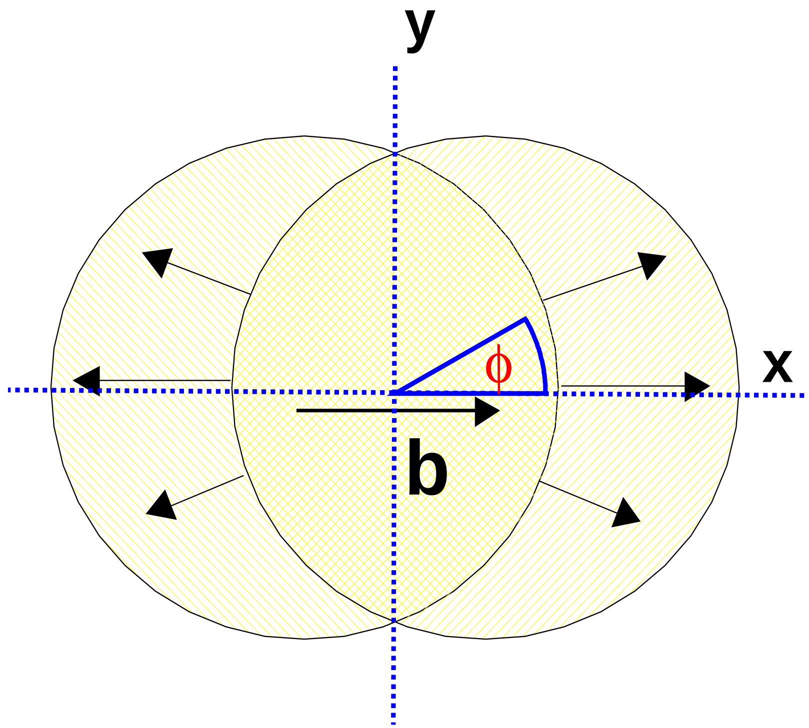
$$\langle \xi_i(t)\xi_j(t')\rangle = \frac{\kappa}{3}\delta_{ij}\,\delta(t-t')\,.$$

- κ is the mean squared momentum transfer per unit time.
- $1/\eta_D$ is what we intuitively called τ_R^{charm}.
- The fluctuation dissipation theorem relates the noise to the drag:

$$\eta_D = \frac{\kappa}{2TE}$$

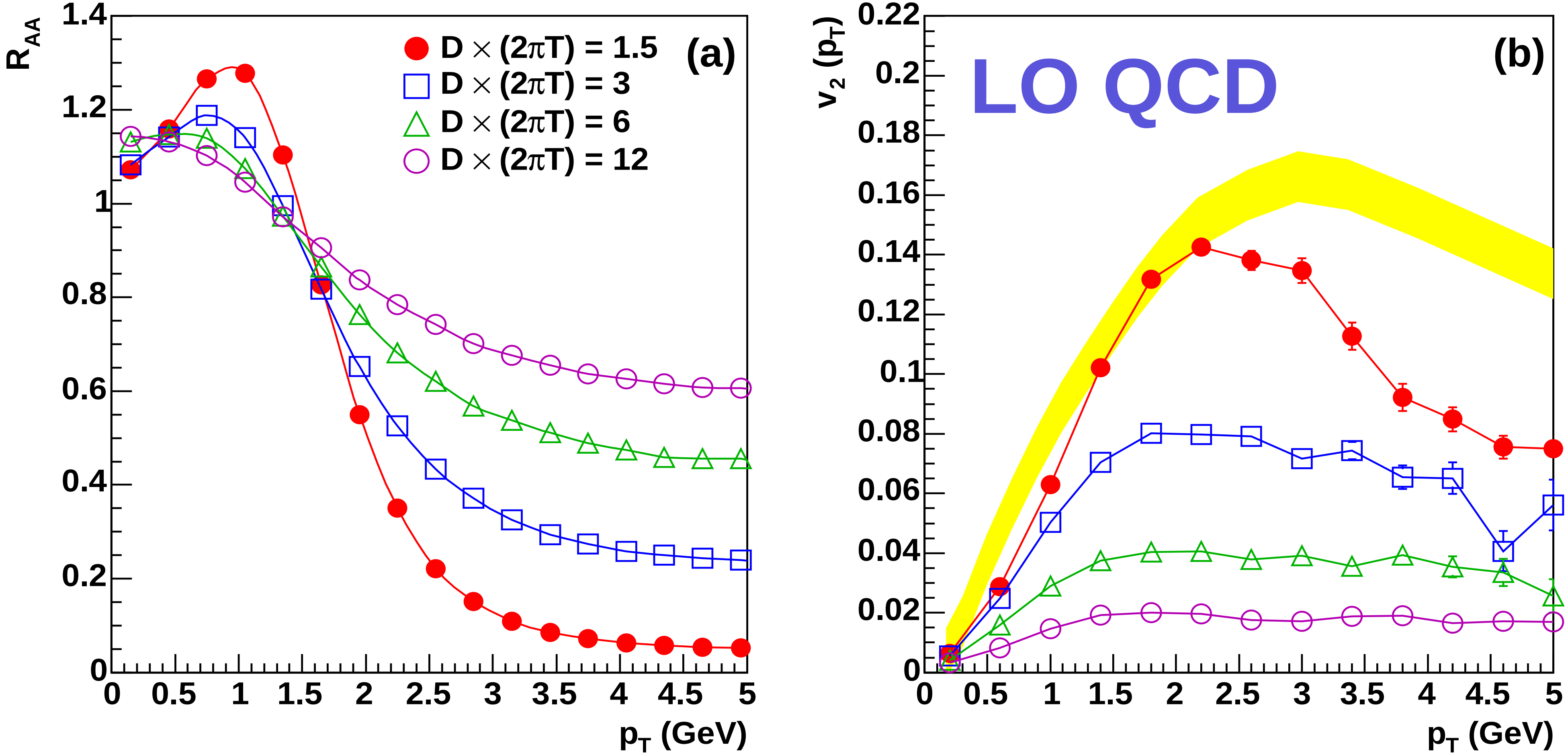
Hydro + Heavy Quarks





- Put the heavy quarks into the hydro subject to Drag + Langevin Random Kicks
- Take ideal EOS $p = e/3$ and a Bjorken Expansion
- Take initial spectrum of heavy quarks from LO-pQCD.

Results for R_{AA} and v_2 for charm quarks:



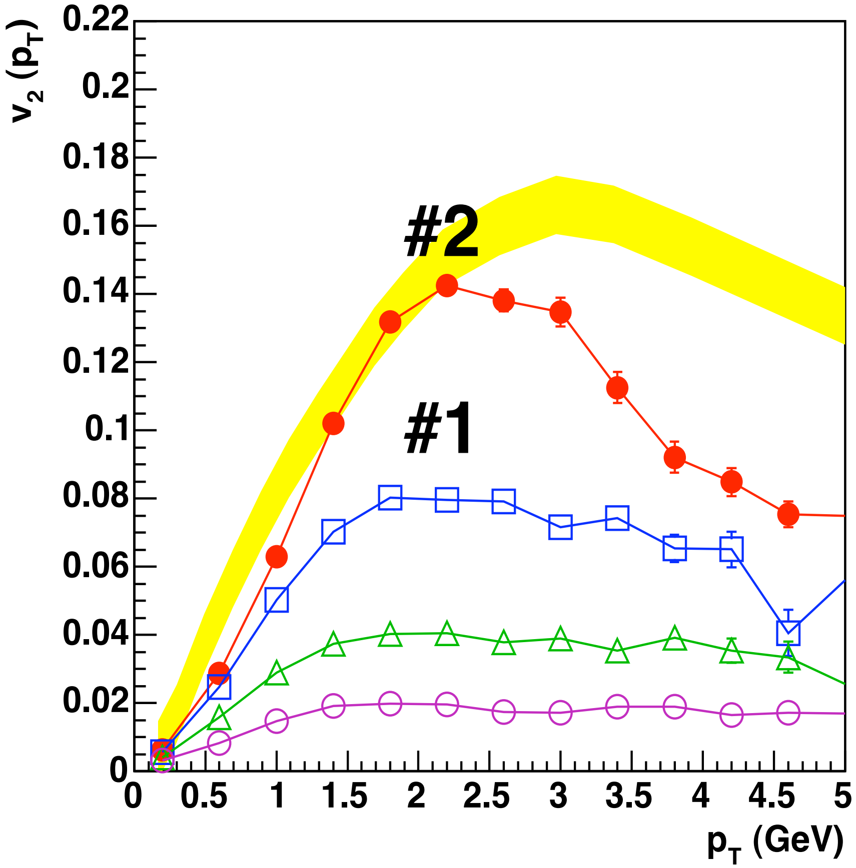
- No significant suppression until

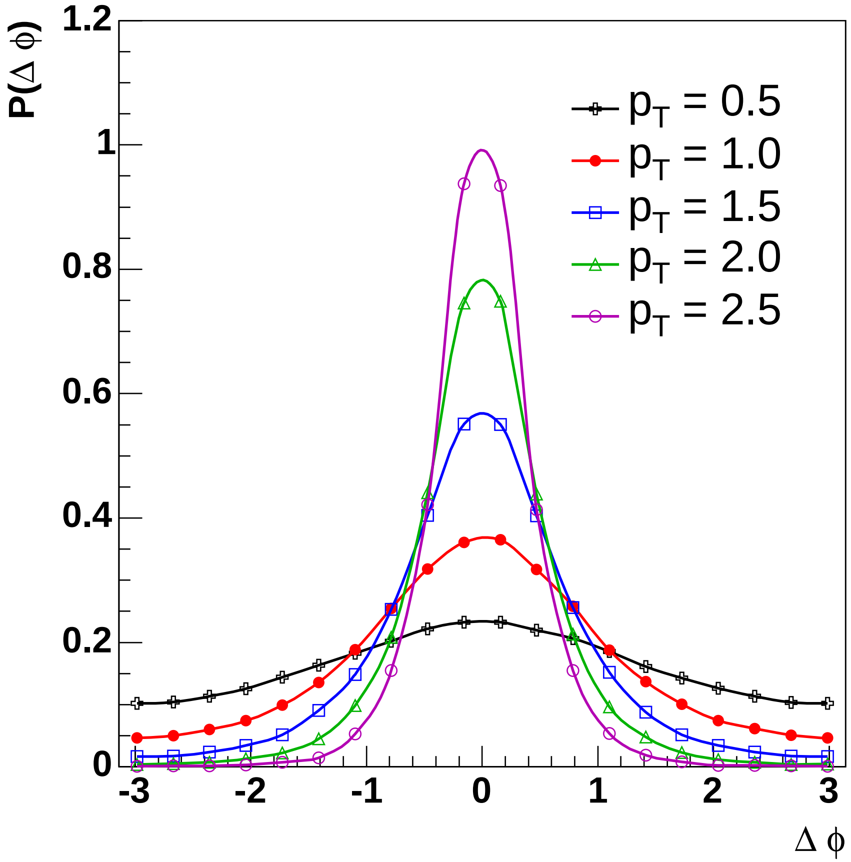
$$D \approx 12 \times \frac{1}{2\pi T} \quad \text{remember} \quad D = \frac{6}{2\pi T}\left(\frac{0.5}{\alpha_s}\right)^2$$

Transition from hydro-like to kinetic regime #1

Examine the initial-angle final-angle correlation function in #1

$$P(\Delta\phi) = \text{ Probability the angle changes by } \Delta\phi$$

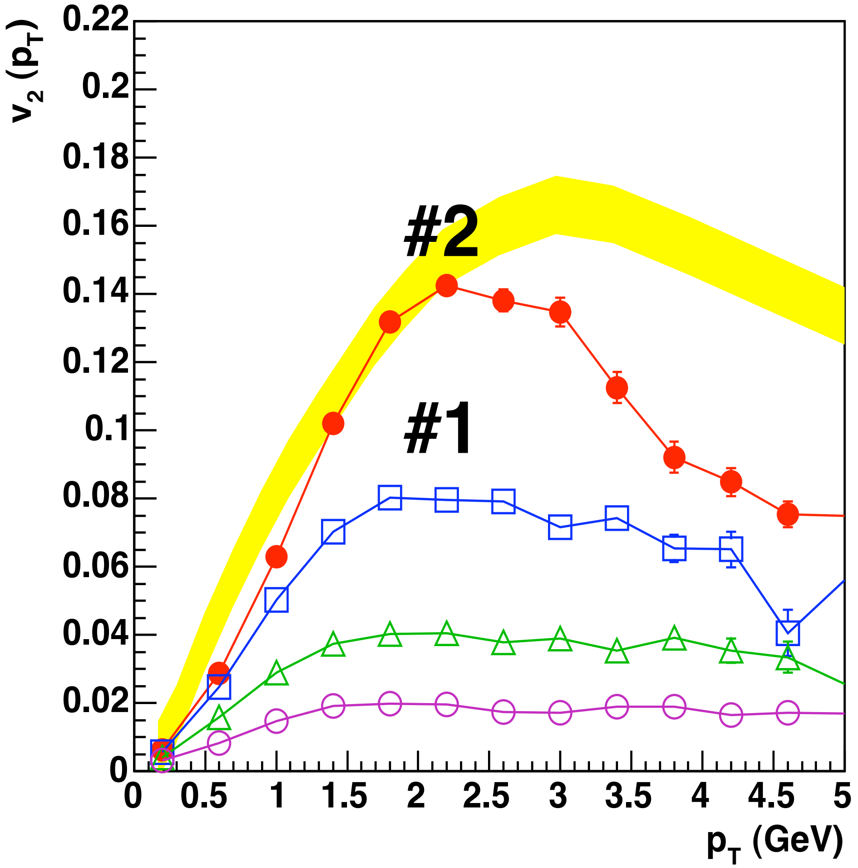


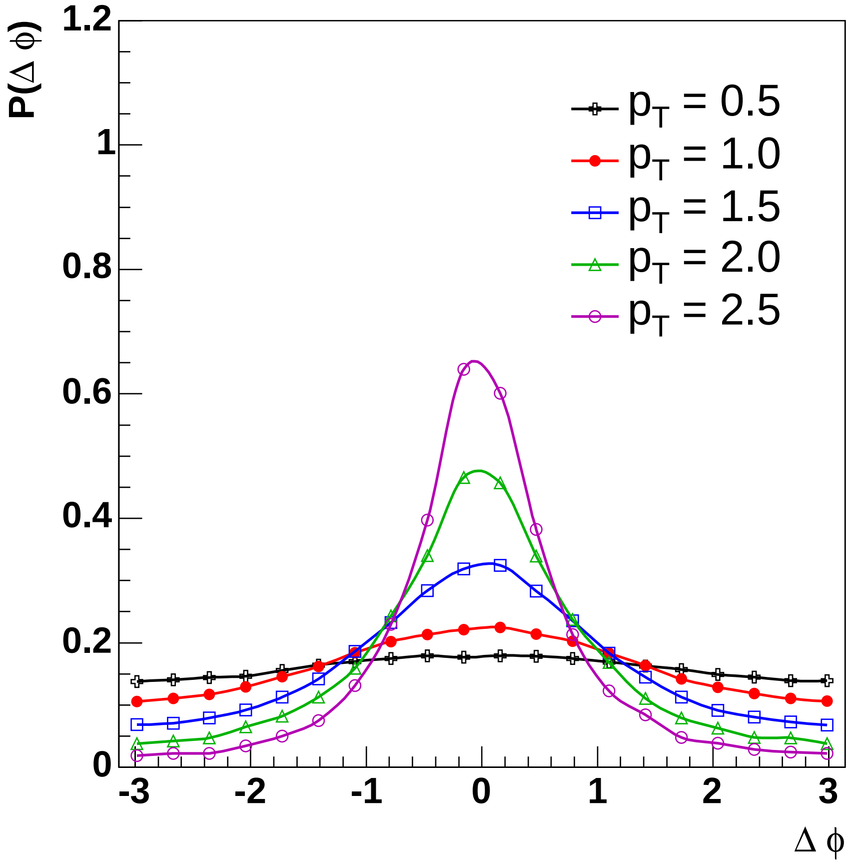


Transition from hydro-like to kinetic regime #2

Examine the initial-angle final-angle correlation function in #2

$$P(\Delta\phi) = \text{ Probability the angle changes by } \Delta\phi$$





Conclusions

- Hydro is Qualitatively Correct as a function of centrality and p_T
- Definite failures in peripheral collisions.
- Need a formalism which interpolates between equilbirium and kinetics to describe $v_2(p_T)$ and R_{AA}
- The transport scale needed to describe $v_2(p_T)$ (without quark coalesence) is too small to describe R_{AA}

Solving the Relativistic Navier Stokes Equations RNSE

- The RNSE as written can not be solved. There are unstable modes which propagate faster than the speed of light.

- Why? Because the stress RNSE tensor is not allowed time to change.

$$T^{ij}_{vis}\Big|_{\text{instantly}} = \eta\left(\partial^i v^j + \partial^j v^i - \frac{2}{3}\delta^{ij}\partial_i v^i\right)$$

- Can make many models (at least seven) which relax to the RNSE. (Drude, Maxwell, P.C. Martin, Mueller, Israel, L. Lindblom, R. Geroch, Ottinger)

$$T^{ij}_{vis}\Big|_{\omega\to 0} \sim \eta\left(\partial^i v^j + \partial^j v^i - \frac{2}{3}\delta^{ij}\partial_i v^i\right)$$

- In the regime of validity of hydrodynamics the models all agree with each other and with RNSE.

Can solve these models

A simple model: Inspired by H.C. Ottinger, Physica 1997

- Imagine a tensor c_{ij} which relaxes quickly to $\partial_i v_j + \partial_j v_i$

$$\partial_t c_{ij} - (\partial_i v_j + \partial_j v_i) = \frac{\bar{c}_{ij}}{\tau_0} + \frac{\langle c_{ij} \rangle}{\tau_2}$$

where $\bar{c}_{ij} = (tr\,\mathbf{c})\,\delta_{ij}$ and $\langle c_{ij} \rangle = c_{ij} - \frac{1}{3}\bar{c}_{ij}$

- For small τ_0 and τ_2 we have:

$$c_{ij} \approx \tau_0 \delta_{ij}\, \partial_i v^i + \tau_2(\partial_i v_j + \partial_j v_i - \frac{2}{3}\delta_{ij}\partial_l v^l)$$

- Then the “effective” pressure for small strains is given by:

$$T_{ij} \approx p(\delta_{ij} - a_1\, c_{ij})$$

Compare this to the canonical form:

$$T_{ij} \approx p\delta_{ij} + \sigma\partial_i v^i + \eta(\partial_i v_j + \partial_j v_i - \frac{2}{3}\delta_{ij}\partial_l v^l)$$

Can map, $(\tau_0, \tau_2, a_1) \rightarrow (\sigma, \eta, c_\infty)$

Another Model: (Inspired by Lindblom and Geroch, Phys. Dev. D1994)

- Write a set conservation/balance laws:

$$\begin{aligned} \partial_\mu(N^\mu) &= 0 \\ \partial_\mu(T^{\mu\nu}) &= 0 \\ \partial_\mu(A^{\mu\alpha\beta}) &= I^{\alpha\beta} \end{aligned}$$

where

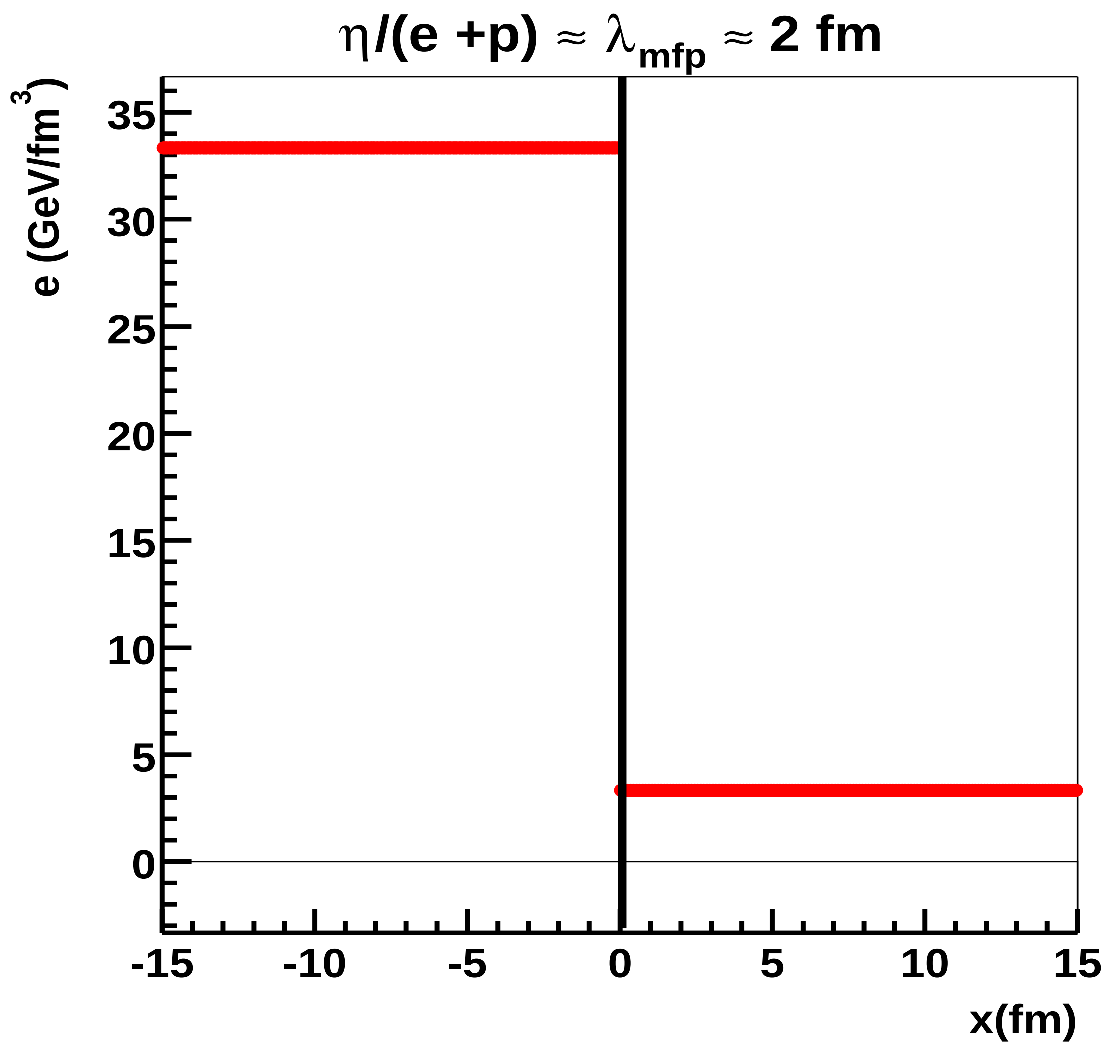
$$\begin{aligned} N^\mu &= n u^\mu \\ T^{\mu\nu} &= e u^\mu u^\nu + p\Delta^{\mu\nu} + u^\mu q^\nu + u^\nu q^\mu + \tau^{\mu\nu} \\ A^{\mu\alpha\beta} &= 2T\Delta^{\mu(\alpha}u^{\beta)} \\ I^{\alpha\beta} &= -\frac{T}{\eta}\tau^{\alpha\beta} - \frac{2T}{3\sigma}\Delta^{\alpha\beta} - \frac{2T}{\kappa T}\left(q^\alpha u^\beta + q^\beta u^\alpha\right) \end{aligned}$$

- A completely different model at short times

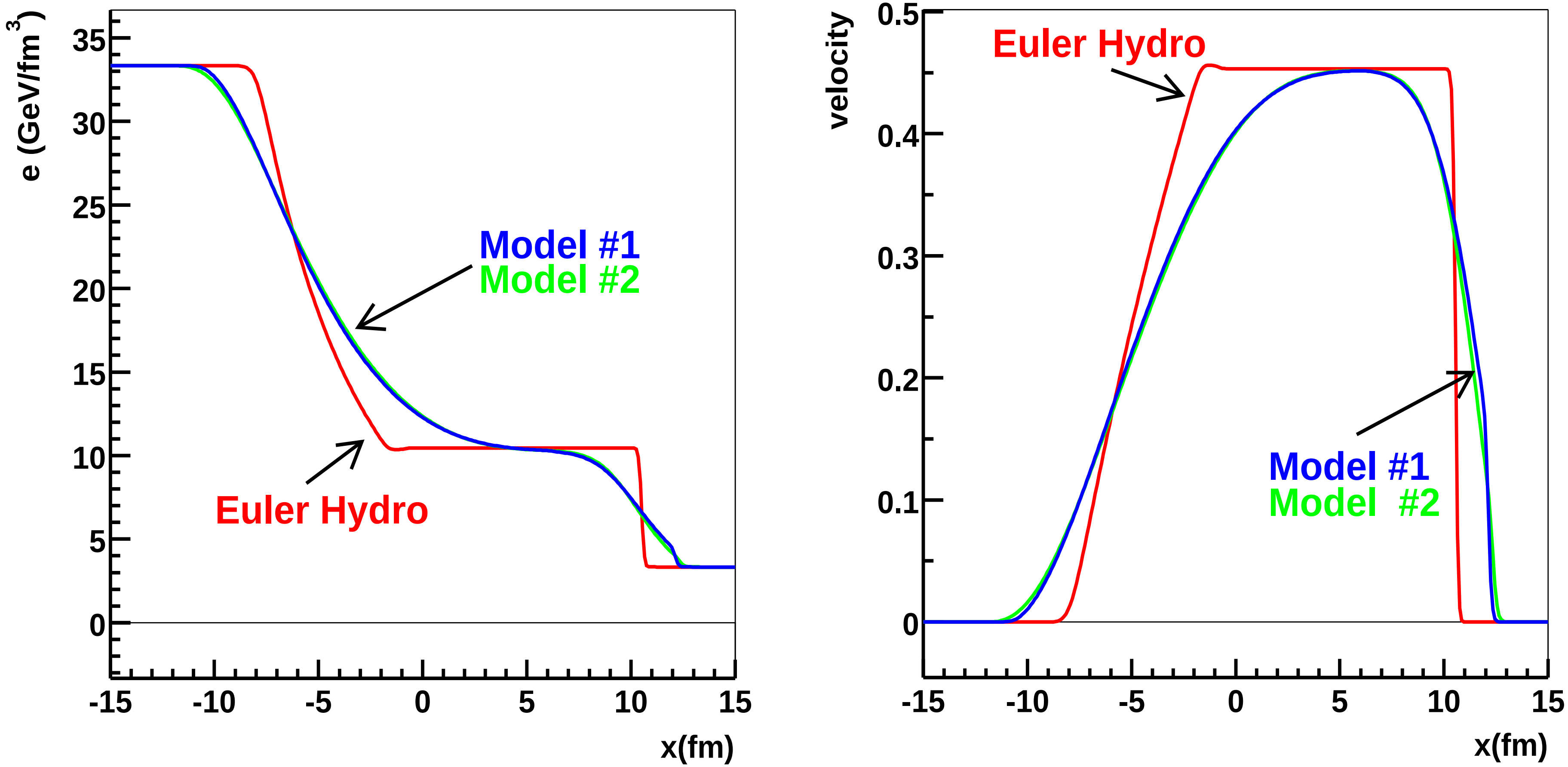
- Only the long time behavior is the same. The long time behavior is controlled by the viscous coefficients.

None of the details of these models should matter.

Sod’s Test Problem

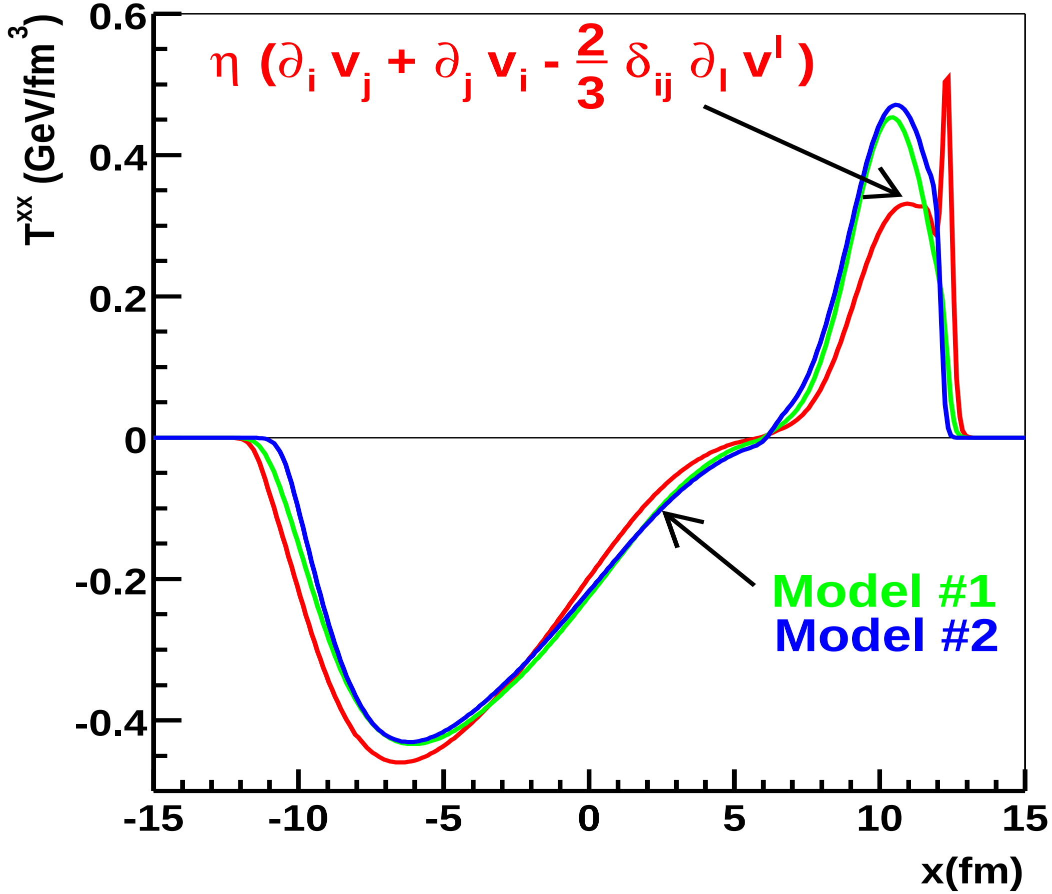


Compare the different models:



The solutions are very similar but different from ideal hydro.

Compare the stress tensor with the Navier Stokes Equations:



The stress tensor is close to its canoncial form.

Summary & Warnings

- All models agree about the solution to the Navier Stokes equations
- The stress energy tensor is almost always very close to

$$T^{ij} \sim \eta \left(\partial^i v^j + \partial^j v^i - \frac{2}{3} \delta^{ij} \partial_l v^l \right)$$

Warnings: This holds in the <u>regime of validity</u> of hydrodynamics.

1. The only natural initial condition is

$$T^{ij}|_{\tau_0} = \eta \left(\partial^i v^j + \partial^j v^i - \frac{2}{3} \delta^{ij} \partial_l v^l \right)$$

2. In general the models have several free parameters. In the regime of validity the solution only depends on the viscosity . Check this!

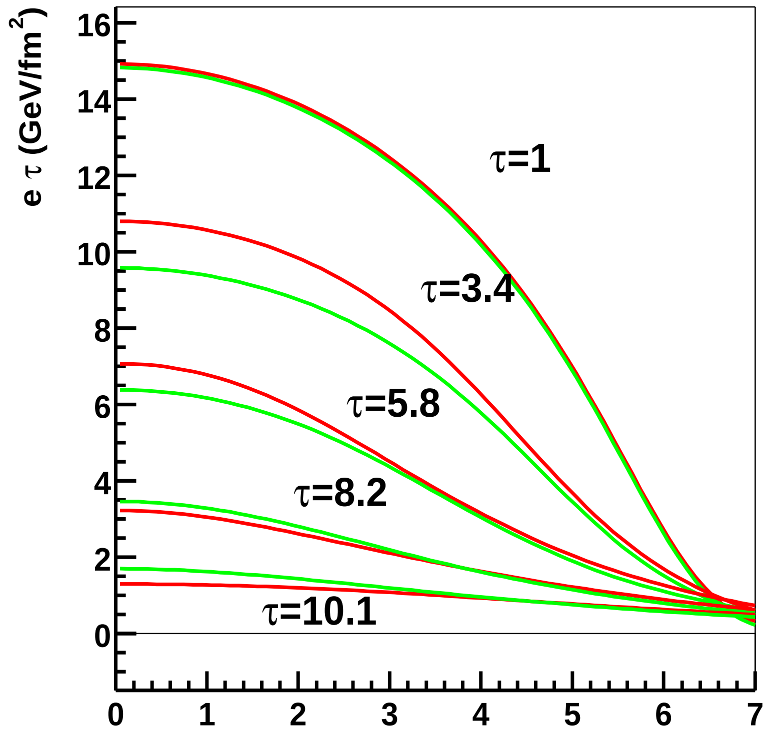
3. Werner-Israel becomes acausal away from equilibrium states.

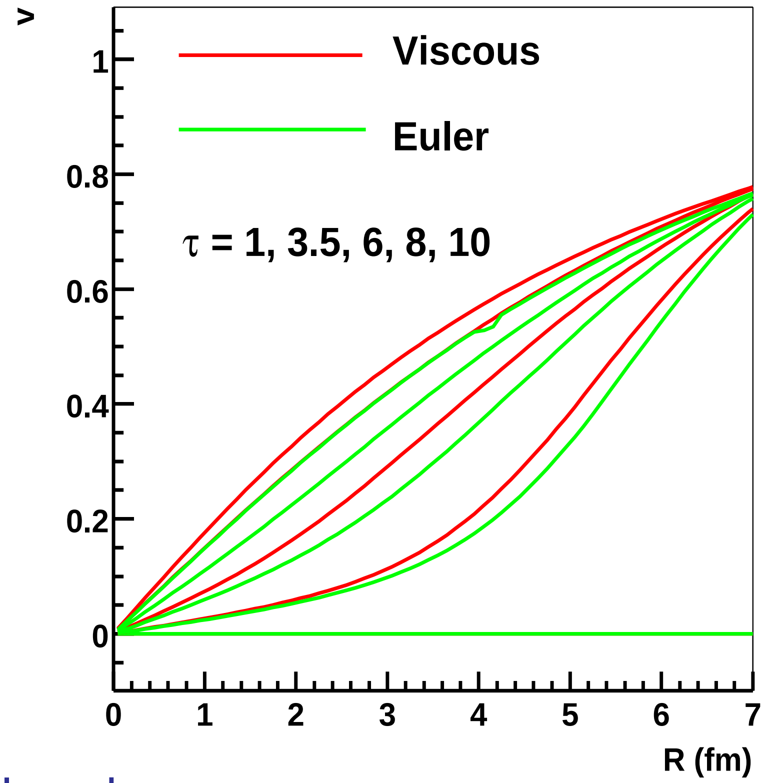
When the viscous term is about half of the pressure :

- The models disagree with each other.
- T^{ij} is not asymptotic with $\sim \eta(\partial^i v^j + \partial^j v^i - \frac{2}{3}\delta^{ij}\partial_l v^l)$

Freezeout is not arbitrary but is signaled by the equations

Bjorken Solution with transverse expansion:

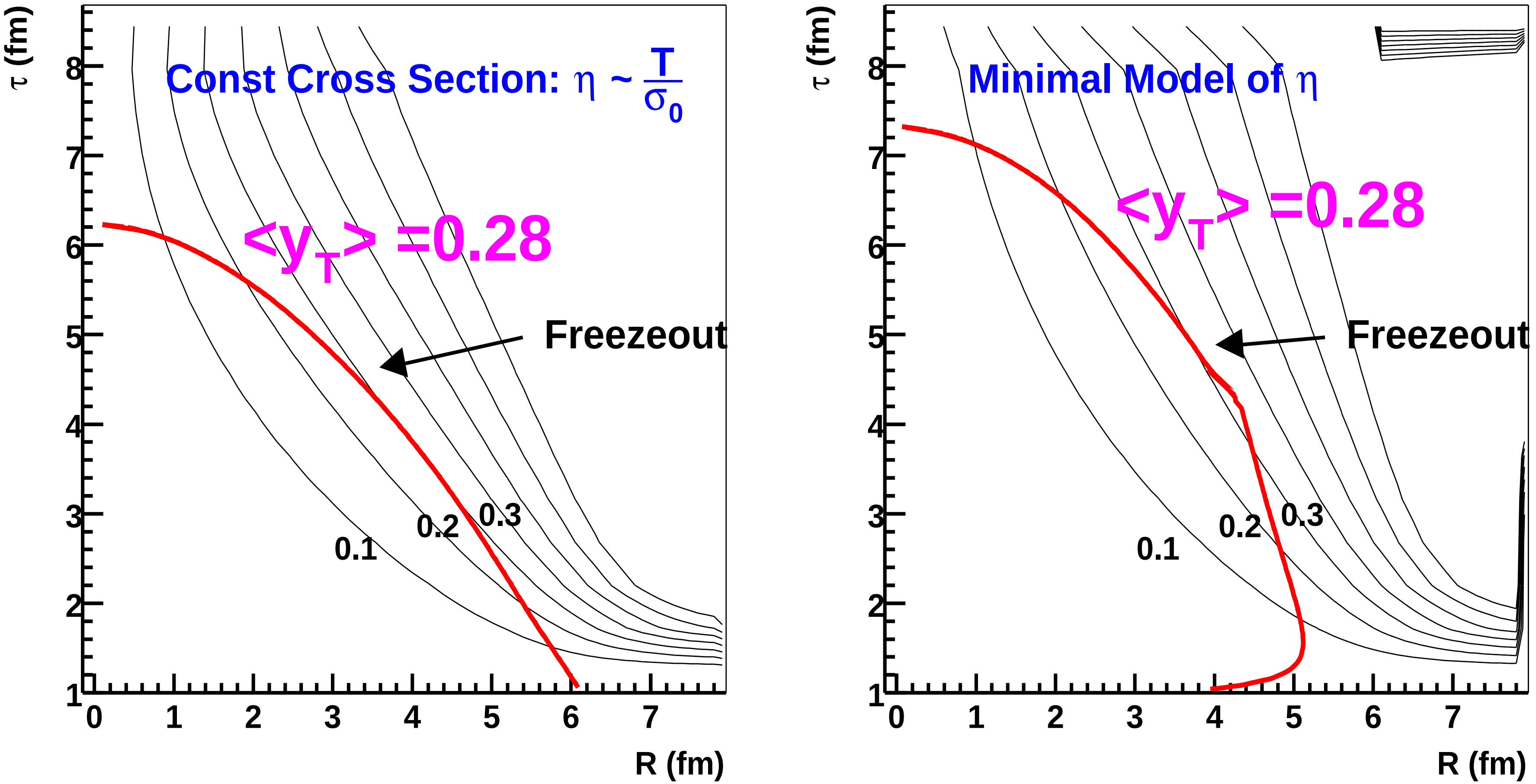




- First the viscous case does less longitudinal work.
- Then the transverse velocity grows more rapidly because the transverse pressure is larger.
- The larger transverse velocity then reduces the energy density more quickly than ideal hydro.

Viscous corrections do NOT integrate to give an O(1) change to the flow.

Compare the two models of viscosity:



The minimal model of η and the Const X.-section model have the same radial flow.

Conclusions:

- Viscosity does not change the ideal hydrodynamic solution particularly much.